Journal of Statistical Physics, Vol. 38, Nos. 5/6, 1985

Renormalization-Group Derivation of Navier–Stokes Equation

P. B. Visscher[1]

Received June 26, 1984

The Navier–Stokes equation is proved from first principles (rotational symmetry and conservation of momentum, mass, and energy) using renormalization-group ideas. That is, we consider a system described by one (classical) conserved vector field and two conserved scalar fields, and demonstrate that on a large scale it obeys the Navier–Stokes equation. No assumptions about the physical meanings of the fields are required; in particular, no results from thermodynamics are used. The result comes about because the Euler equation is an exact fixed point of an appropriate scale-coarsening transformation, and the coefficients of the eigenvectors of the transformation with the largest ("most relevant") eigenvalues include (in dimension $d>2$) the thermal conductivity and the bulk and shear viscosities, leading to Navier–Stokes behavior on a large scale. For $d<2$, the largest eigenvalue corresponds to a convection term, and the Navier–Stokes equation is incorrect. Our method differs from previous renormalization approaches in using time-coarsening as well as space-coarsening transformations. This allows renormalization trajectories to be determined exactly, and allows the determination of the macroscopic behavior of specific microscopic models. The Navier–Stokes equation we obtain is almost, but not exactly, the same as the conventional one; distinguishing between them experimentally would require measurement of the very small asymmetry of the Brillouin line in a simple fluid.

KEY WORDS: Renormalization group; transport equations; Navier–Stokes equation.

1. INTRODUCTION

The Navier–Stokes equation has been used for over a century as a phenomenological description of the macroscopic behavior of fluids. Arguments for the universal applicability of this description have long been

[1] Department of Physics and Astronomy, University of Alabama, University, Alabama 35486.

given,[1] based on writing the lowest-order differential operators which have the correct symmetry. There are two basic difficulties with these arguments:

(A) How does one define a density field in a fluid of discrete particles, as a continuous function of position?

(B) The arguments apply just as well to a two-dimensional fluid as to a three-dimensional fluid. But within the last 15 years, it has become apparent that the conclusion is incorrect for a two-dimensional fluid, which does *not* obey the Navier–Stokes equation.

Difficulty (A), "coarse-graining," has been approached in many different ways. One can smooth out a delta-function density by smearing it over many interatomic spacings[2] and examining only the large-wavelength ($k \to 0$) limit or let the atoms become infinitely small,[3] or consider a limit in which a correlation time approaches zero.[4] The approach we adopt is to use a discrete formulation of hydrodynamics which was introduced several years ago.[5] This avoids the necessity of dealing with continuous functions at all, until one is near the fixed point (i.e., looking at the system on a very large time scale). Then, as we will see later, the cells can be taken to be much smaller than the physically relevant lengths and the discrete description merges naturally into a continuum description.

Difficulty (B) is related to the well-known "long-time tail" problem. The velocity autocorrelation function decays at long times as $t^{-d/2}$, where d is the spatial dimensionality.[6] The shear stress autocorrelation behaves similarly, and its time integral is proportional to the shear viscosity via the Green–Kubo formula.[7] But the integral diverges for $d = 2$, so it appears that one cannot define a viscosity. This conclusion is supported by molecular-dynamics simulation,[8–10] which indicates that a two-dimensional fluid does not obey the Navier–Stokes equation. Even in three dimensions, these effects lead to difficulties such as slow convergence of the Green–Kubo integral, strong (power-law) system-size dependence of the viscosity obtained in nonequilibrium simulations, and nonanalytic strain-rate dependence of the shear stress.[11]

The reason for the long-time tails and the failure of the Navier–Stokes equation in two dimensions has become clear; it is due to the nonlinear convective term [$(\mathbf{u} \cdot \nabla)\mathbf{u}$ in $d\mathbf{u}/dt$], which becomes increasingly important on large scales. This conclusion has emerged from both mode-coupling[12] and approximate renormalization-group[13] calculations. In spite of the fact that the qualitative physics involved is understood, it has not previously proved possible to make a precise renormalization-group analysis of this problem which displays exactly the fixed point and the eigenvalues and eigenvectors of the linearized renormalization transformation. This failure is related to the development of non-Markovian ("memory") behavior as the space scale is coarsened. This appears to us to be inevitable as long as one insists on retaining a small *time* scale. In the work of Forster, Nelson, and Stephen[13] on the case of an incompressible fluid, for example, the use of a continuous time variable (i.e., an infinitesimally small time scale) leads to a non-Markovian renormalized equation of motion. It is possible in principle to cope with this by an expansion in powers of frequency,[13] and the zero-frequency limit considered in Ref. 13 gives the right fixed point (i.e., the higher orders in frequency are "irrelevant" in the critical-phenomena sense.) However, we are here principally interested in the corrections to the fixed point, and it does not appear to be possible to compute the eigenvectors and eigenvalues exactly by this approach. We will show that the problems can be resolved by the use of a *time*-coarsening as well as a space-coarsening transformation: the system then remains Markovian (in the sense that future behavior depends on only a few past times) and can be exactly followed as it approaches the fixed point. This method has been applied previously to disordered diffusive systems,[14] in which it was possible to exactly compute nonanalytic long-time tails arising from nonlinear effects very similar to those in a fluid. This suggests that the method should be able to calculate long-time tails and related effects in fluids.

In addition to allowing more exact study of fixed points than continuous-time methods, our discrete scale-coarsening transformations can be applied directly to small-scale equations of motion (EOMs) obtained by molecular-dynamics simulation. Such EOMs have already been calculated numerically,[15] but prior to the present exact analysis of the approach to the fixed point it was not possible to determine macroscopic properties accurately.

We will begin by briefly describing the discrete formulation of hydrodynamics (Section 2) and a parameterization of the discrete equations of motion which simplifies application of the scale-coarsening transformations (Section 3). We find only two suitable transformations, the fixed points of which are exactly the two most common approximations to the Navier–Stokes equation: the Euler equation (Section 4) and the incompressible-fluid equation (Section 7). By including the most relevant eigenvectors near a fixed point, we get the Navier–Stokes equation if $d > 2$, together with an exact description of the most relevant fluctuations (Section 6).

This amounts to a first-principles derivation of the linearized Navier–Stokes equation assuming only conservation of mass, energy, and momentum. Concepts from thermodynamics which must be arbitrarily assumed in conventional derivations, such as pressure, entropy, temperature, and "local equilibrium," are not assumed here. Quantities equivalent to pressure, entropy, and temperature emerge in a natural way from the renormalization-group analysis, however, suggesting that an analysis of this sort may perhaps provide an attractive pedagogical alternative to the introduction of these concepts through axiomatic thermodynamics.

2. DISCRETE HYDRODYNAMICS

The application of discrete hydrodynamics to fluids is very similar to its application to the calculation of long-time tails in disordered diffusive systems.[16,14] We will summarize here the discrete formulation, which is described in detail in Ref. 16. For any given distance scale Δr and time scale Δt, we describe the state of the system at times t_{ph} (integer multiples of Δt) by the mass, energy, and momentum contents of cubical cells of length dr, denoted $c^{\alpha}_{\text{ph}}(r_{\text{ph}}, t_{\text{ph}})$. The subscript ph means "physical," as opposed to dimensionless; the index α is M, E, P_x, P_y, P_y, or P_z for the mass, energy, and momentum contents, respectively, and r_{ph} is the position vector of the center of a cell. For purposes of scale coarsening, we construct an equation of motion in terms of dimensionless contents

$$c^{\alpha}(r, t) = c^{\alpha}_{\text{ph}}(r\Delta r, t\Delta t)/\Delta c^{\alpha} \tag{2.1}$$

where r is the dimensionless position vector of the center of a cell of unit length, the integer t is a dimensionless time, and Δc^M, Δc^E, and Δc^P are conveniently chosen units of mass, energy, and momentum.

The results of any macroscopic experiment can be predicted from these discrete variables, if the scales Δr and Δt are chosen appropriately. The dynamics of the system (i.e., the joint probability distribution of all the variables) can be completely specified by giving the equilibrium time-correlation functions (mean values of arbitrary products). In a stationary system, this is equivalent[5] to specifying the probability distribution for the contents at time Δt, in a constrained ensemble in which the contents at time $t \leqslant 0$ have fixed values. We will refer to this distribution as the "discrete equation of motion." Our concern in this paper is mostly with Markovian EOMs in which there is no dependence on times $t < 0$. This distribution can also be described by its moments. The first moments are the mean values of the contents $c^{\alpha}(r, 1)$ at time Δt in an ensemble having fixed constents $c(r, 0)$, $c(r, -1)$,... at times $t \leqslant 0$. Such a moment is distinguished by square brackets [ ], and expanded as a power series in the $c(r, 0)$s:

$$[c^\alpha(r,1)] = \sum_{r',\alpha'} [c^\alpha(r,1)]_{c^{\alpha'}(r',0)}\, c^{\alpha'}(r',0)$$
$$+ \sum_{r',\alpha'} \sum_{r'',\alpha''} [c^\alpha(r,1)]_{c^{\alpha'}(r',0)c^{\alpha''}(r'',0)}\, c^{\alpha'}(r',0)\, c^{\alpha''}(r'',0) + \cdots \qquad (2.2)$$

The square brackets with c^αs as subscripts are just constants, the coefficients in the power series. To achieve rapid convergence near some nominal values c_0^α of the contents, we have redefined c^α by subtracting c_0^α. It follows that there is no constant term; $[c^\alpha(r,1)]=0$ when $c^\alpha(r,0)=0$.

The corresponding equation for the second moment does have a constant term, for which we use a subscript 1 (formally, the product of zero c^αs):

$$[c^\alpha(r,1)\, c^{\alpha'}(r',1)] = [c^\alpha(r,1)\, c^{\alpha'}(r',1)]_1 + \cdots \qquad (2.3)$$

The coefficient $[cc]_1$ describes the conditional fluctuations of the contents.

The coefficients in Eqs. (2.2) and (2.3) will be referred to as discrete propagators. They depend only on differences of the rs, and can be dealt with most conveniently in terms of Fourier transforms[14] such as $G_{\alpha'}^\alpha$, defined by

$$(2\pi)^d\, \delta(k-k')\, G_{\alpha'}^\alpha(k,1;k',0) = \sum_{r,r'} e^{-ikr+ik'r'} [c^\alpha(r,1)]_{c^{\alpha'}(r',0)} \qquad (2.4)$$

This describes the effect of quantity α' (i.e., M, E, or $\mathbf{P}$) at $t=0$ on quantity α at $t=1$. It is periodic in the ks because of the discreteness of the rs. Because the other arguments are redundant, we will refer to it as $G_{\alpha'}^\alpha(k)$.

The Fourier transform (FT) of the conditional correlation, or fluctuation propagator, Eq. (2.3), is denoted $G^{\alpha\alpha'}(k)$. We will also use the nonlinear propagator $[c^\alpha]_{c^{\alpha'}c^{\alpha''}}$ of Eq. (2.2), whose FT is $G_{\alpha'\alpha''}^\alpha(k;k',k'')$ (here $k'+k''=k$). We will regard an equation of motion E as being specified by the Gs defined above.

We will want to perform space- and time-coarsening transformations S and T, and a mass, energy, and momentum rescaling transformation[2] R, on an equation of motion E which describes a system on the scale Δr, Δt, Δc^E, Δc^M, Δc^P. We define S, T, and R by requiring that SE describe the same system on the scale $2\Delta r$, Δt, Δc^M, Δc^E, Δc^P, that TE describe it on the scale in which $2\Delta t$ replaces Δt, and that RE do this when $2\Delta c^\alpha$ replaces Δc^α. The action of these transformations on the propagators G is derived in detail in

[2] More generally, we could rescale each conserved quantity separately. However, we show in the Appendix that one can only get a nontrivial fixed point if they are rescaled by the same factor, as they are by the above R.

Ref. 16. It is very simple in the limit of small Δr, which suffices for our present purposes: for example, the space-coarsened $G^{\alpha}_{\alpha'}$ and $G^{\alpha\alpha'}$ are

$$G^{\alpha}_{\alpha'}(k)' = G^{\alpha}_{\alpha'}(k/2) \tag{2.5}$$

$$G^{\alpha\alpha'}(k)' = 2^d G^{\alpha\alpha'}(k/2) \tag{2.6}$$

3. PARAMETERIZATION OF THE DISCRETE EOM'S: EXPONENTIAL EOM'S

We will use a particular parameterization of the space of discrete equations of motion which makes it easy to apply the rescaling transformations. We define a set of "exponential EOMs,[3] obtained by exponentiating (in a sense made precise below) "infinitesimal generator" equations of motion E_g. The exponentiation is done in the continuous-space ($\Delta r \to 0$) limit, in which the propagators are functions of continuous variables r; this allows us to parametrize the propagators of the generating EOM as power series in a Fourier variable k. Once the continuous-space exponential EOM has been defined, we can obtain finite-Δr discrete EOMs by space coarsening.[14] On an intuitive level, the generator EOMs can be thought of as evolving the system through an infinitesimal time, and being related to an equation of motion in a conventional continuous-time theory. For example, if a propagator of E_g is $G^{\alpha}_{\alpha'}(k)$, the equation of motion for the kth Fourier component $c^{\alpha}(k, t)$ of conserved quantity α in a continuous-time theory would be

$$(d/dt)\, c^{\alpha}(k, t) = \sum G^{\alpha}_{\alpha'}(k)\, c^{\alpha'}(k, t) \tag{3.1}$$

But we will use the generating equation of motion here *only* as a mathematical device; Eq. (3.1) does *not* make sense in the present context, because our t is discrete.

The exponentiation procedure is described in detail in Ref. 14: we will summarize it here. The finite-time exponential EOM is defined from the generating EOM (which is assumed Markovian) by

$$E = \exp(E_g) \equiv \lim_{n \to \infty} (\mathbb{1}_E + E_g/n)^n \tag{3.2}$$

(here $\mathbb{1}_E$ is the identity EOM, which assigns to all variables at $t = 1$ the same values they had at $t = 0$). This requires defining the sum and product of discrete-time EOMs. The sum is just defined by adding the

[3] In Refs. 14 and 16, we referred to this set as the "invariant manifold."

corresponding propagators. The product is defined by a graphical procedure described in detail in Ref. 14. Basically the product $E_b E_a$ evolves a system for one time interval according to E_a, and then for another accoding to E_b. If $E_a = E_b$, this is just the same as time coarsening: $TE = EE$. The virtue of parametrizing the EOM by the generating propagators G_g is that they transform very simply under the time-coarsening transformation T:

$$T: G_g \to 2G_g \tag{3.3}$$

The actions of S and R on G_g are the same as on G [Eqs. (2.5) and (2.6)].

We will assume in this paper that the fixed points we are looking for can be obtained as exponential EOMs; this requires in particular that they are Markovian. This is certainly true on a large scale in a fluid. As we will see in Section 6, long-time tails and viscosity renormalization, which are sometimes thought of as involving non-Markovian effects, are nonetheless correctly treated by our procedure. We can also account for truly non-Markovian effects such as viscoelasticity in this formulation, as we discuss in Section 8.

4. SEARCHING FOR FIXED POINTS

Let us look among our exponential EOMs for a fluid system for one which is a fixed point under a suitable coarsening transformation (a combination of S, T, and R). So as not to prejudice the matter, we consider an arbitrary product ST^zR^y.

To do this, consider the most general form of the linear propagator $G^{\alpha}_{\alpha'}$ for a system with one vector and two scalar αs (i.e., a fluid). We suppose here that $G^{\alpha}_{\alpha'}$ is an infinitesimal generator, although the symmetry requirements are the same for the finite-time propagators. Spatial isotropy implies that a scalar propagator like $G^M_M(\mathbf{k})$ must be a function of k^2 only, which we expand in a power series

$$G^M_M(\mathbf{k}) = D_{MM}k^2 + O(k^4) \tag{4.1}$$

[There is no constant term since conservation of mass requires $G^M_M(0) = 0$.] In analogy with the diffusion case,[16] we may think of D_{MM} as the diffusivity of mass density. In searching for a fixed point under the scale-coarsening transformation ST^zR^y, we can ignore the k^4 terms: they are less "relevant" in the critical-phenomena sense. That is, the coefficient of such a term shrinks under coarsening faster than D_{MM} does, and will be zero at the fixed point and smaller than D_{MM} near the fixed point. To see this, note that the eigenvalue of D_{MM} under S is 2^{-2} (i.e., S: $D \to 2^{-2}D$, from

Eq. (2.5)] and that of the coefficient of k^4 is 2^{-4}. (Their respective eigenvalues under $T^z R^y$ are unknown yet, but are the same.) Similar considerations for G_E^M, G_M^E, and G_E^E lead to D_{ME}, D_{EM}, and D_{EE}. A scalar–vector propagator like G_{Pj}^M ($j = x$, y, or z) is forced by isotropy to be k_j times a function of k^2, which can be expanded

$$G_{Pj}^M(\mathbf{k}) = -ik_j v_{MP} + O(k^3) \tag{4.2}$$

where again the k^3 term is less relevant. The reality of the real-space propagators implies that the Ds and vs are real. We define v_{PM}, v_{EP}, and v_{PE} similarly.[4] Finally, isotropy allows the vector–vector propagator to have terms in the dyadic $k_j k_l$ and in $k^2 \delta_{jl}$ (it also must vanish at $k = 0$), and we write it

$$G_{Pj}^{Pl}(\mathbf{k}) = -D_T k^2(\delta_{jl} - \hat{k}_j \hat{k}_l) - D_L k_j k_l + O(k^4) \tag{4.3}$$

We may think of D_T and D_L as the diffusivities of transverse and longitudinal momentum, respectively.

Thus the most general linear generating propagator may be written in matrix form to order k^2 as

$$G_{\alpha'}^{\alpha}(\mathbf{k}) = \begin{pmatrix} -D_{MM}k^2 & -D_{ME}k^2 & -iv_{MP}\mathbf{k} \\ -D_{EM}k^2 & -D_{EE}k^2 & -iv_{EP}\mathbf{k} \\ -iv_{PM}\mathbf{k} & -iv_{PE}\mathbf{k} & -D_T k^2(\mathbb{1} - \hat{k}\hat{k}) - D_{PP}\mathbf{k}\mathbf{k} \end{pmatrix} \tag{4.4}$$

where $\mathbb{1}$ is the identity 3×3 matrix and $\hat{k} = \mathbf{k}/|k|$.

If we use this generating propagator in the conventional time-derivative equation (3.1), we get something which looks very like the phenomenological Navier–Stokes equation,[17] with the correspondences

$$D_T = (\Delta t/\Delta r^2)(\eta/\rho), \qquad D_L = (\Delta t/\Delta r^2)[\xi + (4\eta/3)]/\rho \tag{4.5}$$

where η and ξ are the shear and bulk viscosities.

The D_{MM} and D_{ME} terms in Eq. (4.4) correspond to ∇^2 terms in the $d\rho/dt$ equation, which are conventionally omitted from the phenomenological theory. The argument for this is based on identifying the mass flux with the momentum density, which is certainly valid in a uniform system (i.e., to order k^0). The question really hinges on its validity to order

[4] We can relate the vs to thermodynamic quantities by comparing Eqs. (3.1) and (4.2) to the equation of continuity and the ordinary Euler equation. This gives $v_{MP} = \Delta c_P \Delta t/\Delta c_M \Delta r$, $v_{EP} = (\Delta c_P \Delta t/\Delta c_E \Delta r)$ $\partial e(\rho, s)/\partial \rho$, $v_{PM} = (\Delta c_M \Delta t/\Delta c_P \Delta r)$ $\partial p(\rho, e)/\partial \rho$, and $v_{PE} = (\Delta c_E \Delta t/\Delta c_P \Delta r)$ $\partial p(\rho, e)/\partial e$, where ρ is the mass density, e is the energy density, p is the pressure, and s is the entropy per unit mass.

k^1, however, which is not at all obvious. The experimental consequences of nonzero D_{MM} or D_{ME} appear not to be easily measurable; see Section 5.

The de/dt equation gives

$$D_{EM} = \Delta t\lambda\, \partial T(\rho, e)/\partial\rho \qquad \text{and} \qquad D_{EE} = \Delta t\lambda\, \partial T(\rho, e)/\partial e \tag{4.6}$$

where λ is the thermal conductivity and T the temperature. In the present nonthermodynamic context D_{EM} and D_{EE} can be thought of as *defining* the temperature as a linear combination of c^E and c^M, up to a constant factor.

Let us now write the most general form for the conditional-correlation generator $G^{\alpha\alpha'}(\mathbf{k})$. The symmetry considerations are exactly the same as for the linear generator $G^{\alpha}_{\alpha'}$ except that the real-space correlation [Eq. (2.3)] must be symmetric, so $G^{\alpha\alpha'}(k) = G^{\alpha'\alpha}(k)^*$, and the most general matrix is

$$G^{\alpha\alpha'}(\mathbf{k}) = \begin{pmatrix} a_{MM}k^2 & a_{ME}k^2 & ib_M\mathbf{k} \\ a_{ME}k^2 & a_{EE}k^2 & ib_E\mathbf{k} \\ -ib_M\mathbf{k} & -ib_E\mathbf{k} & a_T k^2(1-\hat{k}\hat{k}) + a_{PP}\mathbf{k}\mathbf{k} \end{pmatrix} \tag{4.7}$$

where the as and bs are real.

We must now determine how the Ds, vs, as, and bs transform under S, T, and R, from Eqs. (2.5), (2.6), and (3.3). Writing D for $D^{\alpha}_{\alpha'}$, etc., we get

$$S\colon\quad D \to 2^{-2}D, \quad v \to 2^{-1}v, \quad a \to 2^{d-2}a, \quad b \to 2^{d-1}b \tag{4.8}$$

$$T\colon\quad D \to 2D, \quad v \to 2v, \quad a \to 2a, \quad b \to 2b \tag{4.9}$$

$$R\colon\quad D \to D, \quad v \to v, \quad a \to 2^{-2}a, \quad b \to 2^{-2}b \tag{4.10}$$

From these we can compute the eigenvalues under the combined transformation ST^zR^y. First consider the fluctuation generator:

$$ST^zR^y\colon\quad a \to 2^{d-2+z-2y}a, \quad b \to 2^{d-1+z-2y}b \tag{4.11}$$

If any b is nonzero, it is more relevant than the as, and the fixed point will have $a = 0$. But the correlation matrix [Eq. (4.7)] with $a = 0$ and $b \neq 0$ violates the basic requirement of nonnegativity [i.e., $f^*_\alpha G^{\alpha\alpha'}(k) f_{\alpha'} \geqslant 0$ for any column vector f; this follows from the requirement that the square of an arbitrary real linear combination of c^αs be positive]. Thus we must conclude[5] that $b = 0$.

To determine z, we must consider how the parameters of the linear propagator behave under the coarsening transformation:

$$ST^zR^y\colon\qquad D \to 2^{z-2}D, \quad v \to 2^{z-1}v \tag{4.12}$$

[5] Note that this is *not* a requirement of symmetry; in particular, a less relevant term $ib'k^2\mathbf{k}$ would be perfectly allowable.

Clearly at a nontrivial fixed point one of these parameters should be finite (neither zero nor infinite). There are evidently two choices:

(1) The "ballistic" coarsening transformation: choosing $z=1$ makes velocities invariant (v has eigenvalue 1) so v is finite but $D=0$ at the fixed points; we will call these the "Euler fixed points."

(2) The "diffusive" coarsening transformation: choosing $z=2$ gives finite Ds and $v \to \infty$ at the "incompressible-fluid" fixed points (Section 7.)

Here we will examine the Euler fixed points, for which Eqs. (4.11) and (4.12) become

$$STR^{y}: \quad D \to 2^{-1}D, \quad v \to v, \quad a \to 2^{d-1-2y}a \tag{4.13}$$

Choosing a value for the exponent y is to some extent a matter of taste; it just determines the content scale we use for the fluctuations. In the diffusive case,[16] it was natural to choose y so that the most relevant conditional fluctuations are finite at the fixed point; this also yielded finite equilibrium fluctuations at the fixed point. In the Euler case, this corresponds to $y=(d-1)/2$ (so the as have eigenvalue 1 and are finite at the fixed point). This turns out to be awkward; the fixed point would have nine parameters (four vs and five as). It would also have divergent equilibium fluctuations: as we will see in Section 6, the fluctuations are generated at rates proportional to the as, and die away at rates proportional to the Ds, so the equilibrium fluctuation is proportional to a/D. If the Ds are zero, the fluctuations of an initially constrained system simply increase linearly with time without bound. A much more aesthetic choice is to require a and D to have the same eigenvalue, so the equilibrium fluctuations have eigenvalue 1. This gives $y=d/2$, and only the vs are nonzero at the fixed point. The fixed-point EOM has no fluctuation generator. It can describe the evolution of a system with arbitrary fluctuations, but those fluctuations will never change.

Assuming $y=d/2$, then, we have found the generators of a four-parameter family of fixed points of the transformation $STR^{d/2}$, one for each choice of the four vs. The EOM of any fluid system must approach one of these Euler fixed points; the values of the four parameters will be related to the thermodynamic properties of the fluid.

We now want to compute the actual (finite-time) fixed-point EOM E by exponentiating one of these generating EOMs, say, E_g. We will first calculate the linear propagator matrix [Eq. (2.4)] of E. The rules given in Section 3 for adding and multiplying EOMs amount, for the case of the linear propagator matrices, to adding and multiplying matrices; therefore, the exponentiation defined by Eq. (3.2) reduces to conventional matrix exponentiation. We are thus faced with the task of exponentiating the fixed point generating propagator matrix [Eq. (4.4) with $D=0$]:

$$G_{\alpha'}^{\alpha}(k) = \begin{pmatrix} 0 & 0 & -iv_{MP}\mathbf{k} \\ 0 & 0 & -iv_{EP}\mathbf{k} \\ -iv_{PM}\mathbf{k} & -iv_{PE}\mathbf{k} & 0 \end{pmatrix} \tag{4.14}$$

Clearly the algebra would be easier if this matrix had only two nonzero entries. We can achieve this by exploiting the mathematical equivalence between the scalar contents c_M and c_E of mass and energy: the equations of motion allowed by symmetry would be the same if they were replaced by any two linear combinations, say, c_p and c_s. There are four degrees of freedom in such a transformation, of which two are just normalization constants: we will exploit the two remaining degrees to decouple one of the new variables from the momentum content c_P, to order k. It is not hard to see that this is achieved by using as new variables

$$\begin{pmatrix} c_p \\ c_s \\ vc_P \end{pmatrix} \equiv \begin{pmatrix} v_{PM} & v_{PE} & 0 \\ -v_{EP} & v_{MP} & 0 \\ 0 & 0 & v \end{pmatrix} \begin{pmatrix} c_M \\ c_E \\ c_P \end{pmatrix} \equiv Q \begin{pmatrix} c_M \\ c_E \\ c_P \end{pmatrix} \tag{4.15}$$

Here $v^2 \equiv v_{MP}v_{PM} + v_{PE}v_{EP}$; we rescale the momentum content by v for mathematical convenience [it makes Eq. (4.16) symmetric to order k]. The new dimensionless variables c_p and c_s turn out to be related to the pressure and the entropy.[6] We will see [Eq. (4.21)] that v is the dimensionless sound velocity.

The linear propagator for c_p and c_s is $\tilde{G} = QGQ^{-1}$, where G is the propagator for c_M and c_E [Eq. (4.14)] and Q is the transformation matrix in Eq. (4.15):

$$\tilde{G} = \begin{pmatrix} 0 & 0 & -iv\mathbf{k} \\ 0 & 0 & 0 \\ -iv\mathbf{k} & 0 & 0 \end{pmatrix} \equiv -ivK \tag{4.16}$$

Equation (4.16) defines a matrix K; with the vector parts written out in detail for $d=2$ it is

$$K = \begin{pmatrix} 0 & 0 & k_x & k_y \\ 0 & 0 & 0 & 0 \\ k_x & 0 & 0 & 0 \\ k_y & 0 & 0 & 0 \end{pmatrix} \tag{4.17}$$

[6] To be precise, the relations in footnote 4 imply that $(\Delta c_P/\Delta t\, \Delta r^2)c_p$ is the excess over the nominal pressure, and $[\Delta c_M\, \Delta c_E\, \Delta c_P\, \Delta r^2\, \Delta t\; \partial e(\rho, s)/\partial s]^{-1} c_s$ is the excess entropy per unit mass.

Then in the same two representations

$$K^2 = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{kk} \end{pmatrix} = \begin{pmatrix} k^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_x k_x & k_y k_x \\ 0 & 0 & k_x k_y & k_y k_y \end{pmatrix} \tag{4.18}$$

If we define

$$\hat{K} = K/k \tag{4.19}$$

then $\hat{K}^3 = \hat{K}$ and $\hat{K}^2\hat{K}^2 = \hat{K}^2$, i.e., $\hat{K}^2$ is a projection operator onto the longitudinal sound modes.

It should now be clear why we refer to this as the "Euler" fixed point: if we use the propagator (4.16) in the conventional time-derivative equation (3.1), we get exactly the usual Euler equation.

It is easy to show that the actual finite-time propagator $\exp[\tilde{G}(k)]$ is

$$\exp(-ivK) = (\mathbb{1} - \hat{K}^2) + \hat{K}^2 \cos vk - i\hat{K} \sin vk \tag{4.20}$$

In matrix form, the propagator is

$$\exp(-ivK) = \begin{pmatrix} \cos vk & 0 & -i\hat{k} \sin vk \\ 0 & 1 & 0 \\ -i\hat{k} \sin vk & 0 & \mathbb{1} - \hat{k}\hat{k}(\cos vk - 1) \end{pmatrix} \tag{4.21}$$

This propagator can be easily Fourier transformed into real space in one dimension, where it has a simple physical interpretation: Defining $x = r - r'$, we get

$$G(x) = \begin{pmatrix} \delta(x-v) + \delta(x+v) & 0 & \delta(x-v) - \delta(x+v) \\ 0 & \delta(x) & 0 \\ \delta(x-v) - \delta(x+v) & 0 & \delta(x-v) + \delta(x+v) \end{pmatrix} \tag{4.22}$$

An initial condition with excess pressure at $r' = 0$ at $t' = 0$ can be evolved to $t = 1$ by applying this matrix to the appropriate column vector:

$$G \begin{pmatrix} c_P = 1 \\ c_s = 0 \\ c_P = 0 \end{pmatrix} = \begin{pmatrix} \delta(x-v) + \delta(x+v) \\ 0 \\ \delta(x-v) - \delta(x+v) \end{pmatrix} \tag{4.23}$$

Thus two sound pulses are produced, moving to the right and left with dimensionless velocity v; the rightward one has positive momentum, and the leftward one a negative momentum. An initial momentum at $r'=0$ produces the same rightward pulse but a leftward pulse of opposite phase; an initial entropy excess at $r'=0$ just stays where it is. A transverse wave (with $c_{\mathbf{P}}$ perpendicular to $\mathbf{k}$), which of course can exist only for $d>1$, is also unchanged. Thus the "Euler" fixed point we have found has the behavior of the conventional Euler equation, and describes an inviscid, thermally insulating fluid.

What we have shown by our change of variables is that any of the four-parameter family of Euler fixed points can be transformed into one of a one-parameter family, parametrized by the sound speed v. It is worth reiterating the great qualitative difference between the Euler fixed point in a fluid and the Fick's law fixed point in a diffusive systems,[16] or the incompressible fixed point in a fluid. In either of the latter, any initial correlations will relax with time into uniquely defined equilibrium correlations. At the Euler fixed point, fluctuations neither grow nor decay. This is, of course, related to the much greater importance of states far from equilibrium (e.g., turbulent states) in fluids than in diffusive systems.

5. THE PROPAGATORS OF THE EXPONENTIAL EOM'S

Let us now move away from the Euler fixed points and calculate the EOM defined by the most general generating EOM, Eqs. (4.4) and (4.7). This will allow us later, by specializing to the EOMs near a particular Euler fixed point, to calculate the eigenvectors and eigenvalues of a linearized coarsening transformation.

As we saw in Section 4, actual computations are easier in the "pressure–entropy" representation than in the "mass–energy" one. Thus we begin by transforming the general generating linear propagator [Eq. (4.4)] by the transformation matrix Q [Eq. (4.15)]:

$$\tilde{G}(k)=QGQ^{-1}=\begin{pmatrix} -D_{pp}k^2 & -D_{ps}k^2 & -ivk \\ -D_{sp}k^2 & -D_s k^2 & 0 \\ -ivk & 0 & -D_T k^2(\mathbb{1}-\hat{k}\hat{k})-D_{PP}\mathbf{kk} \end{pmatrix} \tag{5.1}$$

where D_{pp}, D_{ps}, D_{sp}, and D_s are linear combinations of the old D_{EM}, D_{MM}, D_{EE}, and D_{ME}. As in Section 4, we must exponentiate this matrix to get the finite-time propagators. This is easiest if we write it in terms of projection matrices $\hat{K}^2$ [Eq. (4.19)],

$$\mathbb{1}_T=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-\hat{k}\hat{k} \end{pmatrix}, \qquad \mathbb{1}_S=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2}$$

onto the longitudinal, transverse, and entropy subspaces, respectively. These are a complete set in that the 5×5 unit matrix is

$$\mathbb{1} = \hat{K}^2 + \mathbb{1}_T + \mathbb{1}_S \tag{5.3}$$

With K, these are a set of four commuting matrices. To write Eq. (5.1), we also need three "noncommuting" (with K) matrices

$$\mathbb{1}_{ps} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbb{1}_{sp} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbb{1}_{-} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hat{k}\hat{k} \end{pmatrix} \tag{5.4}$$

Then the generating propagator [Eq. (5.1)] becomes

$$\begin{aligned} \tilde{G}(k) = & -ivK - D_L k^2 \hat{K}^2 - D_T k^2 \mathbb{1}_T - D_s k^2 \mathbb{1}_s \\ & - D_- k^2 \mathbb{1}_- - D_{ps} k^2 \mathbb{1}_{ps} - D_{sp} k^2 \mathbb{1}_{sp} \end{aligned} \tag{5.5}$$

where

$$D_L \equiv (D_{pp} + D_{PP})/2, \qquad D_- \equiv (D_{pp} - D_{PP})/2 \tag{5.6}$$

To get the linear propagator, we need to exponentiate Eq. (5.5). Near the Euler fixed point, there is no reason to make this accurate to higher than first order in the Ds; D^2 has eigenvalue 2^{-2} (i.e., shrinks by this factor under the coarsening transformation $STR^{d/2}$), and is no larger than several other terms in the generator which we have ignored. However, it is easy to exponentiate the commuting terms in v, D_T, D_S, and D_L exactly to all orders, and this will prove useful in discussing the incompressible-fluid fixed point (Section 7), where D_T and D_S are not small. Denoting the sum of these terms by $G_0(k)$, we have

$$\begin{aligned} \exp[G_0(k)] = & \hat{K}^2 \exp(-D_L k^2) \exp(-ivK) + \mathbb{1}_T \exp(-D_T k^2) \\ & + \mathbb{1}_S \exp(-D_S k^2) \end{aligned} \tag{5.7}$$

Thus the "commuting" Ds are the ones which give us the dissipative behavior we associate with the Navier–Stokes equation. Transverse motions are damped by D_T, which is a dimensionless kinematic shear viscosity. Sound waves are damped by D_L, and entropy inhomogeneities are damped by the dimensionless thermal diffusivity D_s. The effects of the noncommuting Ds are much more subtle (in fact, essentially unmeasurable in macroscopic experiments, as we will see later.) We will treat them to first order only, by a graphical procedure derived in detail in Ref. 14 for exponentiating the sum of an "unperturbed" EOM, E, and a small perturbation F. In our case, the unperturbed linear propagator describing E is $G_0(k)$, and that for F is

$$G_F(k) = -D_- k^2 \mathbb{1}_- - D_{ps} k^2 \mathbb{1}_{ps} - D_{sp} k^2 \mathbb{1}_{sp} \tag{5.8}$$

The perturbed finite-time propagator is then

$$\exp[G_0 + G_F] = \exp[G_0] + \int_0^1 du \exp[(1-u)\, G_0(k)]\, G_F \exp[u G_0(k)] \tag{5.9}$$

Equation (5.9) is obtained from the rules of Ref. 14, but is also fairly intuitive: the unperturbed propagator has added to it a term in which the system evolves for dimensionless time u according to G_0, is perturbed by G_F, and evolves again for $1-u$. It is straightforward to evaluate Eq. (5.9), but algebraically messy. We will content ourselves with computing the perturbation only to first order in all the Ds [i.e., using $G_0(k) = -ivK$ only]. Then, for example, we get for the D_{sp} term (using the projection operators $\mathbb{1}_s$ and $\hat{K}^2$ to remind ourselves that $\mathbb{1}_{sp}$ turns pressure into entropy)

$$\begin{aligned} -D_{sp} k^2 \int_0^1 du \exp[-ivK(1-u)]\, \mathbb{1}_s \mathbb{1}_{sp} \hat{K}^2 \exp[-ivKu] \\ = -D_{sp} k^2 \mathbb{1}_{sp} \hat{K}^2 \int_0^1 du \exp[-ivk\hat{K}u] \end{aligned} \tag{5.10}$$

If we restrict it to the pressure-momentum subspace, the matrix G_0 in the exponent is invertible ($\hat{K}^{-1} = \hat{K}$), so $\int du \exp[G_0 u] = G_0^{-1} \exp[G_0 u]$, giving for the term (5.10)

$$-i(D_{sp}/v)\mathbb{1}_{sp} K(e^{ivK} - 1) \tag{5.11}$$

Similar techniques give for the other two terms

$$-i(D_{ps}/v)\, K(e^{-ivK} - 1)\mathbb{1}_{ps} - i(D_-/2v)\, K(e^{-ivK} - e^{ivK})\mathbb{1}_- \tag{5.12}$$

The entire propagator can be written in matrix form using Eq. (4.21),

$$\begin{pmatrix} \begin{matrix}\exp(-D_L k^2)\cos vk \\ -(D_-/v)k \sin vk\end{matrix} & (D_{ps}/v)k \sin vk & i\hat{k}\exp(-D_L k^2)\sin vk \\ (D_{sp}/v)k\sin vk & \exp(-D_s k^2) & i(D_{sp}/v)(1-\cos vk)\mathbf{k} \\ i\hat{k}\exp(-D_L k^2)\sin vk & i(D_{ps}/v)(1-\cos vk)\mathbf{k} & \begin{matrix}[\hat{k}\hat{k}\exp(-D_L k^2)\cos vk \\ +\hat{k}\hat{k}(D_-/v)k\sin vk \\ +(1-\hat{k}\hat{k})\exp(-D_T k^2)]\end{matrix} \end{pmatrix} \tag{5.13}$$

but is more easily used in the form of Eqs. (5.7), and (5.11), (5.12).

Physically, the noncommuting Ds describe pressure–entropy coupling (D_{sp} and D_{ps}) and a time delay or phase lag between the pressure and the momentum in a sound wave [D_-; note that the cos vks on the diagonal of Eq. (5.13) are effectively changed to $\cos(vk \pm D_- k/v)$]. We know of no experimentally measured consequences of them; in the present context, we can attribute this to the extra factor $1/v$ in Eqs. (5.11), (5.12).

We now turn to the fluctuation propagator of the exponential EOM defined by Eqs. (4.4) and (4.7). We must first convert the fluctuation generator [Eq. (4.7)] to the pressure–entropy representation, as we already converted the linear propagator [Eq. (4.4)], to Eq. (5.1). The result is

$$\tilde{G}_g(k) = QG_g(k)Q^{\text{tr}} = \begin{pmatrix} a_{pp}k^2 & a_{ps}k^2 & 0 \\ a_{sp}k^2 & a_s k^2 & 0 \\ 0 & 0 & a_T k^2(1-\hat{k}\hat{k}) + a_{PP}\hat{k}\hat{k} \end{pmatrix} \quad (5.14)$$

where a_{pp}, $a_{sp} = a_{ps}$, and a_s are linear combinations of the old a_{MM}, a_{ME}, and a_{EE}, and Q^{tr} is the transpose of the transformation matrix Q in Eq. (4.15). We compute the fluctuation propagator by the graphical method of Ref. 14, using $E = \exp[G_0]$ again as our "unperturbed" EOM (this has only the "commuting" terms, and no fluctuation propagator) and the fluctuation generator $\tilde{G}_g^{\alpha\alpha'}$ [Eq. (5.14)] as our "perturbation" EOM. The graphical rules[14] give for the fluctuation propagator

$$\tilde{G}^{\beta\beta'}(k) = \int_0^1 du \exp[(1-u)\,G_0(-k)]_\alpha^\beta\, \tilde{G}_g^{\alpha\alpha'}(k) \exp[(1-u)\,G_0(k)]_{\alpha'}^{\beta'} \quad (5.15)$$

(This is exact to all orders in $\tilde{G}_g^{\alpha\alpha'}$, not just to first order: the higher-order terms vanish.) The graph leading to Eq. (5.15) is shown in Fig. 1;

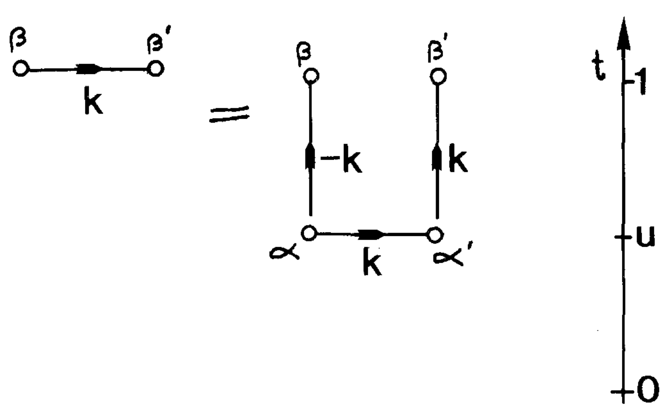

Fig. 1. The graph for calculating the fluctuation propagator $\tilde{G}^{\beta\beta'}(k)$ [Eq. (5.15)].

heuristically the fluctuation at $t=1$ is created by superposing fluctuations (represented by the rightmost horizontal dumbell) generated at different times u, each of which then propagates according to $\exp(G_0)$ for a time $1-u$, as indicated by the vertical lines of length $1-u$. We can write Eq. (5.15) as a matrix product, if we transpose the last factor (otherwise α' and β' are in the wrong order). The part of Eq. (5.15) corresponding to "commuting" terms in $\tilde{G}_g$ is easy to evaluate to all orders in D_L, D_T, D_S, and v; we will evaluate the "noncommuting" part only to zeroth order in the Ds (but exactly in v). The "commuting" part of $\tilde{G}_g$ [Eq. (5.14)] is

$$\tilde{G}_{gc} = a_L k^2 \hat{K}^2 + a_S k^2 \mathbb{1}_S + a_T k^2 \mathbb{1}_T \tag{5.16}$$

[analogous to Eq. (5.5)] where $a_L = (a_{pp} + a_{PP})/2$. It gives (replacing $1-u$ by u)

$$\tilde{G}_c(k) = \int_0^1 du \exp[uM] \tilde{G}_{gc} \tag{5.17}$$

where

$$M = G_0(-k) + G_0(k) = -2[D_L \hat{K}^2 + D_T \mathbb{1}_T + D_S \mathbb{1}_S] k^2 \tag{5.18}$$

(the v term is even in k and cancels). Then

$$M^{-1} = -[\hat{K}^2/D_L + \mathbb{1}_T/D_T + \mathbb{1}_S/D_S]/2k^2 \tag{5.19}$$

so the integral is

$$\begin{aligned}\tilde{G}_c(K) = M^{-1}\tilde{G}_{gc}[\exp(M) - 1] &= (a_L/2D_L)[1 - \exp(-2D_L k^2)]\hat{K}^2 \\ &+ (a_T/2D_T)[1 - \exp(-2D_T k^2)]\mathbb{1}_T \\ &+ (a_S/2D_S)[1 - \exp(-2D_S k^2)]\mathbb{1}_S\end{aligned} \tag{5.20}$$

The noncommuting part of Eq. (5.14) has a term with coefficient $a_- = (a_{pp} - a_{PP})/2$ involving $\mathbb{1}_-$ [Eq. (5.4)], which anticommutes with K, and can therefore be moved past powers of K by replacing K by $-K$:

$$\begin{aligned}\int_0^1 du\, e^{ivKu} a_- k^2 \mathbb{1}_- e^{-ivKu} &= a_- k^2 \int_0^1 du\, e^{2ivKu} \mathbb{1}_- \\ &= -i(a_-/2v)\, K(e^{2ivK} - 1)\mathbb{1}_-\end{aligned} \tag{5.21}$$

Using $\mathbb{1}_{sp} e^{-ivKu} = \mathbb{1}_{sp} \mathbb{1}_s e^{-ivKu} = \mathbb{1}_{sp}$, the a_{ps} term becomes

$$i(a_{ps}/v)[\mathbb{1}_{sp} K(e^{-ivK} - 1) - K(e^{ivK} - 1)\mathbb{1}_{ps}] \tag{5.22}$$

The fluctuation propagator given by Eqs. (5.20)–(5.22) gives the correlations at time Δt, in an ensemble having no fluctuations at time zero.

The equilibrium equal-time averages can be determined from the fluctuation propagator by a self-consistency procedure: We can compute the averages at time Δt by evolving those at $t=0$ via the linear propagator $G_\alpha^\beta(k)$ and adding those generated by the fluctuation propagator. Requiring these averages to be the same as those at $t=0$ gives an equation which can be solved for the averages. This has been done for diffusive systems in Ref. 16, and will be done in detail for the present fluid case in a future publication. To lowest order, the equal-time fluctuation matrix is found to be diagonal in the pressure–entropy–momentum representation:

$$(a_L/2D_L)\hat{K}^2 + (a_S/2D_S)\mathbb{1}_S + (a_T/2D_T)\mathbb{1}_T \tag{5.23}$$

[This can be thought of as the $\Delta t \to \infty$ limit of Eq. (5.20), in which the exponentials vanish.] Equation (5.23) explains the difficulty in defining the fluctuations at the Euler fixed point: they diverge if the Ds are taken to be zero. The numerical values of the fluctuation amplitudes $a_L/2D_L$, $a_S/2D_S$, and $a_T/2D_T$ are, of course, proportional to absolute temperature and can be obtained from thermodynamic results for fluctuations in very large cells.

We have also calculated the unequal-time averages in terms of the as and Ds, and thereby obtained the scattering function $S(k, \omega)$ which is measured in light-scattering experiments; this will also be reported later. The resulting spectrum has the same three terms which are obtained from the phenomenological theory.[18] The first describes an unshifted Lorentzian Rayleigh line (width $\sim D_s$), the second the Brillouin doublet (frequency shift $\sim v$, width $\sim D_L$) and the third an asymmetry in the Brillouin line shape. The relative magnitude of the asymmetry term is a linear combination of D_-/v, D_{sp}/v, D_{ps}/v, and $a_{ps}D_L/a_L$. The only difference between our prediction and the phenomenological one is that in the latter $a_{ps}=0$ and there are fewer independent longitudinal Ds (two, since $D_{MM}=D_{ME}=0$). Thus the two widths D_L and D_S determine everything, and the asymmetry is a particular linear combination of them.[18] In the present theory, the asymmetry is an independent parameter. Thus even a rough measurement of the asymmetry ought to determine which theory is correct. Unfortunately, however, the asymmetry is very small, because of the factor $1/v$. For Ar at 235 K and $\rho = 1\,\mathrm{g/cm^3}$, we have[18] physical parameters $D_{L,\mathrm{ph}} \simeq 10^{-3}\,\mathrm{cm^2/s}$ and $v_{\mathrm{ph}} \simeq 10^5$ cm/s. For scattering wavevector $q \sim 10^5\,\mathrm{cm^{-1}}$, scales appropriate for the Brillouin line are $\Delta r \sim q^{-1} \sim 10^{-5}$ cm, $\Delta t \sim \Delta r^2/D_{L,\mathrm{ph}} \sim 10^{-7}$ s (so the dimensionless linewidth $D_L \sim 1$). Then the dimensionless velocity is $v = v_{\mathrm{ph}}\,\Delta t/\Delta r = 10^3$, and (if $D_- \sim D$) the relative asymmetry is $D_-/v \sim 10^{-3}$. This means that the intensities one halfwidth to the right and left of the line center differ by $\sim 10^{-3}$. The relative differences are much greater farther out in the wings, but clearly this would be a difficult experiment, especially since the resolution of available Fabry–Perot interferometers is no smaller than the linewidth.[19] As far as we are aware, even the linewidth has not been quantitatively measured in a classical simple liquid. The asymmetry may be enhanced by using a larger q, as in neutron scattering. However, it is known[20] that at these wavelengths and frequencies other small-scale effects, such as viscoelasticity, must be taken into account.

6. EIGENVALUES AND EIGENVECTORS: THE NAVIER–STOKES EQUATION

In this section, we will use the explicit formulas derived in Section 5 for the exponential-EOM propagators to determine the most relevant eigenvectors of the linearized coarsening transformation near the Euler fixed point. In fact this is almost trivial, since the coarsening transformation $STR^{d/2}$ [Eq. (4.13)] does not couple v, the Ds, or the as. The change in the exponential EOM to first order in any of these parameters constitutes an eigenvector, whose eigenvalue is determined by the behavior of that parameter under $STR^{d/2}$. The eigenvector corresponding to the sound attenuation coefficient D_L, for example, is seen from Eq. (5.7) to have linear propagator $-K^2e^{-ivK}$, and eigenvalue 2^{-1} [Eq. (4.13)]. Those for the other commuting Ds (D_T and D_S, the shear and entropy diffusivities) have the same eigenvalue; they are "irrelevant" in the critical-phenomena jargon. The eigenvector corresponding to v is "marginal" (has eigenvalue 1); actually there are four of these, corresponding to the original four vs in the mass–energy representation. The eigenvectors for the "noncommuting" D_-, D_{sp}, and D_{ps} have coefficients D/v instead of D [Eqs. (5.11)–(5.12); they have the same eigenvalue 2^{-1} as the commuting Ds with respect to the Euler fixed point, but will behave differently near the incompressible-fluid fixed point (Section 7). All of the other eigenvectors we can obtain by perturbing the linear propagator (e.g., adding Burnett terms of order k^3) have smaller eigenvalues; they are less "relevant." As the scale Δr gets large, they will shrink and the system will be described by an Euler fixed-point EOM plus six terms with coefficients D_L, D_S, D_T, D_-, D_{sp}, and D_{ps}. Thus the Navier–Stokes equation, although originally arrived at phenomenologically, has a very simple renormalization-group interpretation: it is exactly the collection of EOMs represented by the Euler fixed-point EOMs and their most relevant eigenvectors. Any nonpathological EOM for a fluid with the proper symmetry and conservation laws will be led to one of these Navier–Stokes EOMs on a large scale (and on an even larger scale, of course, to the Euler fixed point).

We have thus far only perturbed the linear propagator of the Euler fixed point. We can also turn on the fluctuations (the *a*s). Turning on a_L changes the fluctuation propagator [Eq. (5.20)] by $a_L K^2$ (a sound-wave fluctuation); the eigenvalue is thus again 2^{-1} [Eq. (4.12)], as it is for the other commuting fluctuations a_S and a_T as well. The coefficients of the noncommuting perturbations in Eqs. (5.21), (5.22) are a_-/v and a_{sp}/v, which also give the eigenvalue 2^{-1}. All other perturbations are less relevant. Thus if we take fluctuations into account, there is really an 11-parameter manifold around each Euler fixed point (i.e., 15 parameters in all). Not all of the five *a*s are independent; they are constrained by Galilean invariance.

Let us now consider the nonlinear propagator $G^{\alpha}_{\alpha'\alpha''}(k; k', k'')$, describing the influence of $c^{\alpha'}(k')\, c^{\alpha''}(k'')$ on the content $c^{\alpha}(k)$. The conservation laws require it to vanish when $\mathbf{k} = 0$, so it must have at least one factor of $\mathbf{k}$. Terms with higher powers of $\mathbf{k}$ are less relevant (the eigenvalue has an extra factor 2^{-1} for each factor of k) so we need consider only terms proportional to $\mathbf{k}$, which take the form

$$\tilde{N}^{\alpha}_{\alpha'\alpha''} k_{\alpha} + N^{\alpha}_{\alpha'\alpha''} k_{\alpha'} \tag{6.1}$$

Taking symmetry into account, there are ten independent *N*s, of which seven are convective in nature; it turns out that this means they are expressible in terms of the *v*s because of Galilean invariance. A detailed analysis of them will be given elsewhere. Here we will note only that there are coefficients of the form N^{P}_{PP} which describe convection of momentum. These nonlinear terms will enter the graphical expansion for the linear propagator of the exponential EOM, via the graph of Fig. 2. The effect was calculated explicitly in Ref. 14 for a diffusive system with a nonlinear scattering term. It leads to a long-time tail in the time correlation function and a renormalization of the transport coefficient. The corresponding calculations for the fluid case will be described elsewhere; a long-time tail arises in the momentum-density correlation function, and there is a renormalization of the shear viscosity proportional to $N^2 a_T$, in agreement with the results of Forster, Nelson, and Stephen.[13] The important point here is that in this formulation, a Markovian EOM with a nonlinear generator gives rise in a straightforward way to the "long-time-tail" effects which require the assumption of non-Markovian memory functions in some other formulations. In particular, we can analyze the stability of the Navier–Stokes equation against nonlinear perturbations by examining the eigenvalue of the perturbation described by Eq. (6.1) under the coarsening

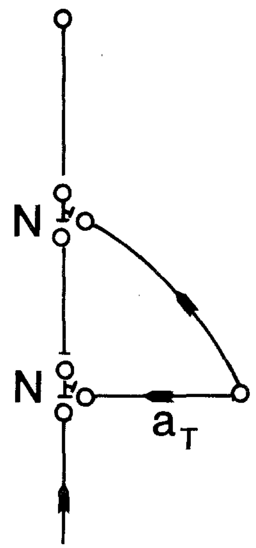

Fig. 2. Graph for renormalizing the linear propagator. The small subgraphs labeled N represent the nonlinear generating propagator [Eq. (6.1)], the horizontal line is the a_T term of the fluctuation generator [Eq. (5.20)], and the other lines represent the unperturbed linear propagator [Eq. (5.7)].

transformation $STR^{d/2}$. We have[16] $S: N \to 2^{-d-1}N$, $T: N \to 2N$ [Eq. (3.3)], and $R: N \to 2N$, so

$$STR^{d/2}: N \to 2^{-d/2}N \tag{6.2}$$

Thus N is less relevant than the Navier–Stokes Ds (eigenvalue 2^{-1}) if $d > 2$, and the Navier–Stokes equation correctly describes a fluid on a large scale. If $d < 2$, it does not; the Ds are less relevant than N, and the macroscopic transport coefficients diverge. If $d = 2$, the transport coefficients diverge logarithmically. Because the parameters of the present formulation can be determined by small-scale molecular-dynamics simulation, it should be possible to calculate the transport coefficient renormalization numerically, which is not possible in mode-coupling approaches.

7. THE INCOMPRESSIBLE-FLUID FIXED POINTS

In looking for fixed points (Section 4), we found two coarsening transformations which had fixed points. One was the ballistic transformation STR^y, which gave the Euler fixed points. The other was the diffusive transformation ST^2R^y, under which $D \to D$, $v \to 2v$, $a \to 2^{d-2y}a$. Thus the fixed points have finite D and infinite v; we may as well choose y

so a is fixed, giving $y = d/2$ just as for the Euler fixed point. Infinite sound velocity amounts to zero compressibility, so we will call these the "incompressible-fluid" fixed points.

Which type of fixed point is most useful in a particular physical problem depends on the characteristic length and time scales of the problem. Given a characteristic physical length Δr, we can define a sound travel time $t_v = \Delta r/v_{\text{ph}}$ and a diffusion time $t_D = \Delta r^2/D_{\text{ph}}$, where typically $t_v \ll t_D$ if Δr is large on a molecular scale. If we want to study phenomena on a short time scale $\Delta t \simeq t_v$ (so $v = v_{\text{ph}}(\Delta t/\Delta r) \simeq 1$ and $D \ll 1$), i.e., sound-related phenomena, we should expand about the Euler fixed point. If we want to study diffusive or viscous phenomena, on the longer time scale $\Delta t \simeq t_D$ (so $v \gg 1$ and $D \simeq 1$), we should expand about the incompressible fixed point.

Computing the propagators of the exponential EOM in the $v \to \infty$ limit is a bit tricky. The largest term in the exponent is $-ivK$; its exponential is given by Eq. (4.20). Roughly speaking, what happens as $v \to \infty$ is that the cosine and sine terms oscillate rapidly and have a negligible effect, so $e^{-ivK} \to 1 - \hat{K}^2$. This is just a projector onto the entropy and transverse-momentum (P_T) subspaces. The generating propagators referring to the longitudinal momentum and pressure subspaces disappear, leaving only the entropy and P_T generators. The propagator of the exponential EOM is exactly

$$\exp(G_g) = \exp(-D_S k^2)\mathbb{1}_S + \exp(-D_T k^2)\mathbb{1}_T \tag{7.1}$$

That is, pressure and longitudinal momentum inhomogeneities at $t = 0$ disappear without a trace, leaving only entropy and transverse momentum at $t = 1$.

The fluctuation propagator at the incompressible-fluid fixed point can be computed just as in Section 5, and is given by Eq. (5.20). Note that there are sound fluctuations, even though the linear propagator instantly banishes them to infinity; we must think of them as being constantly replenished from infinity. This EOM is a function of a_L, D_L, a_T, D_T, a_S, and D_S; there is thus a six-parameter family of incompressible-fluid fixed points.

Unlike the Euler fixed points, these do have well-defined equilibrium states. The procedure described in Section 5 gives Eq. (5.23) for the matrix of equilibrium fluctuations; this was correct only to lowest order in the general Navier–Stokes case, but it is exact at the incompressible fixed point.

The significance of the incompressible-fluid fixed points is that an arbitrary fluid EOM will approach one of them under the diffusive coarsening transformation $ST^2R^{d/2}$. It will exhibit a universal behavior on moderately large scales as the fixed point is approached, which is determined by the eigenvectors of the linearized transformation with the largest eigenvalues. The difficulty in doing this eigenvector analysis is that the general propagator [Eq. (5.13)] does not simply scale when v is doubled by the coarsening transformation (because $\cos vk$ does not). Nor can we expand in a small parameter (such as $1/v$) so that the lowest term does. The most reasonable procedure we have found is to Fourier transform in time [that is, Fourier transform the propagators of an EOM $\exp(E_g u)$ with respect to the dimensionless time u]. Then, for example, the lower left element of the matrix (5.13) becomes

$$G_{Pp}(k, \omega) = \frac{-i\hat{k}D_L(\omega - vk)}{(\omega - vk)^2 + D_L^2 k^4} + (\omega \leftrightarrow -\omega) \tag{7.2}$$

where ω is the dimensionless frequency ($\omega = \omega_{\text{ph}}\, \Delta t$). For large v, this can be expanded in v^{-1}

$$G_{P_p}(k, \omega) = (i\hat{k}D_L/k)v^{-1} + O(v^{-2}) \tag{7.3}$$

Expanding the entire matrix in v^{-1}, we find a piece [including the term in Eq. (7.3)] proportional to $(D_L/v)K$; it is an eigenfunction with eigenvalue 2^{-1} under $ST^2R^{d/2}$, since $v \to 2v$. We find similar eigenfunctions with coefficients (D_{sp}/v) and (D_{ps}/v), with the same eigenvalue. There is also a term with coefficient (D_-/v^2), which is a less relevant eigenfunction with eigenvalue 2^{-2}. The same sort of analysis of the general fluctuation propagator [Eqs. (5.20)–(5.22)] yields eigenvectors with coefficients a_-/v and a_{ps}/v, having eigenvalues 2^{-1}. There are thus a total of five most-relevant eigenvectors, with eigenvalue 2^{-1}, which determine the universal ($d > 2$) large-scale behavior under the diffusive coarsening transformation.

8. CONCLUSION

We have presented what we believe to be the first consistent first-principles derivation of the Navier–Stokes equation which deals with coarse-graining in a precise way and clearly predicts the failure of the equation in dimension $d \leqslant 2$. As a by-product, we obtain a universal description of the fluctuations as well, both in equilibrium and in a constrained ensemble (the latter correspond to the "stochastic" fluctuations of Langevin-type theories).

Because the variables used are contents of discrete cells, the method provides a precise way of connecting small-cell data obtained by molecular-dynamics simulation to macroscopic transport coefficients. This can be done by varying the "bare" parameters (D, v, a) so that the small-cell time correlations calculated from them by the methods of the present paper match those obtained by simulation. The macroscopic transport coefficients can then be calculated as described in Section 6, in a way which explicitly includes the "long-time tail" contribution (due to large-scale eddies) proportional to $N^2 a_T$. The convergence of this method with respect to system size should be much faster than that of Green–Kubo or nonequilibrium methods,[21] which must deal with the large-scale eddies by using a system large enough to contain them. In its present form, our technique must deal with relaxation phenomena such as viscoelasticity and structural, vibrational, or rotational relaxation by using a time interval larger than the relaxation time. However, it is possible to deal with these phenomena more efficiently by including additional dynamical variables (stress, intramolecular energy and angular momentum); this will be discussed in a later publication.

ACKNOWLEDGMENTS

Supported by NSF grant No. CHE-8106122 and by the University of Alabama Research Grants Committee, project No. 1233.

APPENDIX

Instead of defining a single content-rescaling transformation R as in Section 2, we could define a separate mass-rescaling transformation R_M such that is E is the EOM on a scale Δr, Δt, Δc^E, Δc^P, then $R_M E$ is the EOM on a on a scale Δr, Δt, $2\Delta c^M$, Δc^E, Δc^P. Defining R_E and R_P similarly, it is evident that our previous R is just $R_M R_E R_P$. We can now look for a fixed point under a more general transformation than the $ST^z R^y$ of Section 4, namely, $ST^z R_M^{y(M)} R_E^{y(E)} R_P^{y(P)}$. Then v_{MP} scales differently from v_{PM}, in contrast with Eq. (4.10). Equation (4.12) becomes

$$v_{PM} \to 2^{z-1+y(M)-y(P)} v_{PM}, \qquad v_{MP} \to 2^{z-1+y(P)-y(M)} v_{MP} \tag{A1}$$

If $y(M) \neq y(P)$, v_{PM} and v_{MP} canot both be nonzero at the fixed point. And an EOM in which only one is nonzero gives trivial dynamics (physically, the dimensionless sound velocity is zero). For a nontrivial fixed point, we need a pair like v_{MP} and v_{PM} both nonzero. This requires $y(M) = y(P)$ and $z = 1$. For v_{PE} and v_{EP} to remain finite we then need $y(E) = y(P)$. Thus we are led back to the case $y = y(E) = y(P) = y(M)$ considered in the paper.

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